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Dynamical mode locking in commensurate structures with an asymmetric deformable substrate potential

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The overdamped dynamics in the commensurate structures of the one-dimensional Frenkel-Kontorova model subjected to a parametrized deformable periodic substrate potential and driven by a periodic force is examined. It was found that when the shape of the substrate potential starts to deviate from the standard one, new subharmonic steps appear in the response function even in the structures with an integer value of average interparticle distance while the critical depinning force can even decrease for some values of system parameters. These novel phenomena could be particularly relevant for the charge-density wave systems, vortex lattices, and systems of Josephson-junction arrays.

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The dissipative dynamics of driven Frenkel-Kontorova (FK) model has attracted remarkable interest in the recent years. Often referred as one of the simplest and most tractable among many-body models with competing interactions, the standard FK model [1,2] that describes a chain of harmonically interacting particles subjected to an external periodic (sinusoidal) substrate potential provides a deep insight into many physical and biological phenomena. The systems, such as charge-density or spin-density wave conductors [3,4], vortex lattices, and Josephson-junction arrays biased by external currents [5] have been the main impulse for the theoretical studies of dissipative dynamics of the FK model.

In the numerous theoretical and experimental studies, particular interest has been focused on the response of the system to the applied external driver. Using molecular-dynamics simulation [1], Floria *et al.* [2,6] have studied in detail the overdamped dynamics of both the commensurate and incommensurate structures of the one-dimensional standard FK model submitted to the dc and ac forces. For the commensurate structure, they obtained the staircase macroscopic response or the Shapiro steps in the curve for average velocity as a function of average external force $\bar{v}(\bar{F})$. Dynamical mode locking at certain resonant values and quantized increase of the average velocity are the results of the generation of the coherent, time localized and regularly distributed in-time disturbances (instantons) [2,6,7]. In the case of incommensurate structure, the ac-driven dissipative dynamics exhibits the dynamical Aubry transition [6]. The dynamical hull function that describes the driven structure becomes nonanalytical above the transition point and the result of this is the dynamical locking of $\bar{v}(\bar{F})$ at certain resonant values. The mode locking is only possible if the set of the ground state is discrete and it appears to be one of the universal features of the systems with competition of time scale in the ac-driven dynamics.

However, in the real physical systems, the shape of the substrate potential can deviate from the standard (sinusoidal) one and this may affect strongly the transport properties of the system. In the physical situations, such as charge-density waves, Josephson junctions, or crystals with dislocations, the application of standard FK model could be very restricted and it is hard to believe that real physical systems could be "exactly" described by standard models or by employing perturbation methods. Introducing a new family of nonlinear periodic deformable potentials, Remoissenet and Peyrard [8] obtained in a control manner by an adequate choice of parameters a rich variety of deformable potentials related to the physical systems such as Josephson junctions, charge-density wave condensates, and crystals with dislocations. They have shown that the shape of the substrate potential was of great importance for the modeling of discrete systems [8]. The discreteness effects are strongly dependent on the shape of the substrate potential and in the case of a highly deformed potential, even a very large kink could be pinned.

On the contrary, with the large number of studies of the driven standard FK model, a relatively small number of studies have been devoted to the FK model with more realistic potentials. Recently in Ref. [9], transport properties of the continuum system with an asymmetric double-well potential, in particular the stability of asymmetric kinks under thermal forces, have been investigated. Meanwhile, the effect of the deformation of substrate potential on the dynamics of discrete systems, especially on the transport properties of the driven FK model, has not been investigated any further even if it is of great importance, particularly in the systems where discreteness effects are involved.

In the present paper, we will examine the influence of the deformation of substrate potential on dissipative (overdamped) dynamics in different commensurate structures of the FK model driven by dc and ac forces. We consider one from the family of parametrized deformable periodic potentials, the asymmetric deformable potential (ASDP) [8], where by an appropriate choice of the shape parameter, one can move in a controlled manner from the simply periodic

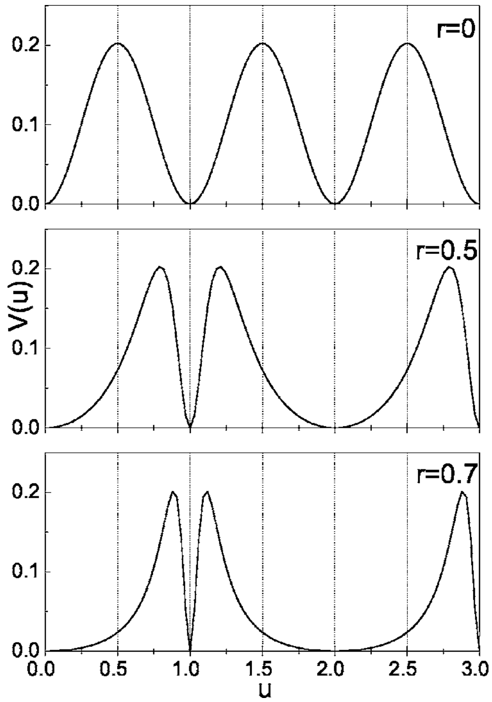

FIG. 1. Substrate potential for $K=4$ and different values of the shape parameter r.

and symmetric sinusoidal (standard) potential to an asymmetric periodic one. We will show that the shape of substrate potential has a strong influence on the system dynamics, in particular, on critical depinning force and dynamical mode-locking phenomena. Deviation from the sinusoidal shape produces the new effects such as appearance of the subharmonic steps even in the commensurate structures with integer values of interparticle distance and decreasing of depinning force for some values of the system parameters. These novel phenomena reported here for the first time cannot exist in the standard models and they result from the deformability of the substrate potential.

We consider the dissipative (overdamped) dynamics of series of coupled harmonics oscillators u_l subjected in an ASDP [8] as follows:

$$V(u) = \frac{K}{(2\pi)^2}\frac{(1-r^2)^2[1-\cos(2\pi u)]}{[1+r^2+2r\cos(\pi u)]^2}, \tag{1}$$

where K is the pinning strength and r is the shape parameter ($-1<r<1$). In Fig. 1, we plotted the ASDP for different values of r. For $r=0$, Eq. (1) reduces in the continuum limit to the standard sine-Gordon (SG) potential, which is integrable. For $0<r<1$, one obtains an asymmetric deformable potential (which is not completely integrable) with a constant barrier height and two inequivalent successive wells with a flat and sharp bottom, respectively. The position ϕ_b of the potential barrier is determined by the relation $\cos(\phi_b/2)=2r/(1+r^2)$. Precisely, here the asymmetry of the substrate potential means that the oscillation frequencies at the two successive potential minima are different.

The system is driven by dc and ac forces: $F(t)=\bar{F}+F_{ac}\cos(2\pi\nu_0 t)$, where $\bar{F}$ is the dc force while F_{ac} and $2\pi\nu_0$ are the amplitude and frequency of ac force, respectively. The equations of motions are

$$\dot{u}_l = u_{l+1} + u_{l-1} - 2u_l - V'(u_l) + F(t), \tag{2}$$

where $l=-\frac{N}{2},\ldots,\frac{N}{2}$. For $r=0$, they reduce to the equations already studied in Refs. [2,6]. Equation (2) has been integrated using the fourth order Runge-Kutta method with the periodic boundary conditions for different commensurate structures with the interparticle average distance (winding number) $\omega=\langle(u_{l+1}-u_l)\rangle$ (ω is rational for the commensurate and irrational for the incommensurate structures). The time step used in the simulations was 0.02τ for lower values of r and 0.0002τ for $r>8$ (τ was the period of ac force). The force was varied with the step 10^{-4} and a time interval of 100τ was used as a relaxation time to allow the system to reach the steady state. The response function $\bar{v}(\bar{F})$ is analyzed by molecular-dynamics simulations for two commensurate structures, $\omega=\frac{1}{2}$ and $\omega=1$, and different shapes of the substrate potential.

If the system is driven by homogenous periodic force, two frequency scales are present in the system: the frequency ν_0 of the external periodic (ac) force and the characteristic frequency of the motion over the periodic substrate potential driven by the average force $\bar{F}$. The competition between these two frequency scales results in the appearance of dynamical mode locking (Shapiro steps) in the unlocking transition at some critical value of the force. In Fig. 2, the average velocity as a function of average driving force obtained for $\omega=\frac{1}{2}$, $K=4$, $F_{ac}=0.2$, $\nu_0=0.2$, and different values of r is presented.

If $K=0$, in the continuum limit, the average velocity is given by $\bar{v}\sim\bar{F}$, while for $K\neq 0$, it locks at certain resonant values and becomes a staircase function.

For the sinusoidal (standard) substrate potential, when $r=0$, in Fig. 2(a), we obtained the well-known result from Ref. [6]. The velocity at every step corresponds to the resonant solution of Eq. (2) and it is given by [2]

$$\bar{v} = \frac{i\omega + j}{m}\nu_0, \tag{3}$$

where i, j, and m are integers. In Fig. 2(a), we can clearly see several harmonic steps corresponding to the value of $m=1$. The subharmonic steps ($m=2$) obtained in Ref. [6] are not visible on the curve for $r=0$ in Fig. 2 due to the too large scale of the axis. The existence of subharmonic steps in the commensurate structures has been matter of many debate. In Ref. [10], it was proved that subharmonic steps do not exist for integer values of ω. For rational, noninteger values of ω, Inui and Doniach [7] and later Falo *et al.* [6] in their works on standard FK model have found the numerical evidence for the existence of subharmonic steps, while for the integer values of ω, they don't exist.

When $r\neq 0$, new substeps appear in the response function and the unlocking transition starts at different values of critical force. In the inset (obtained by rescaling) of Fig. 2(a), we can clearly see the subharmonic steps obtained for $r=0.1$.

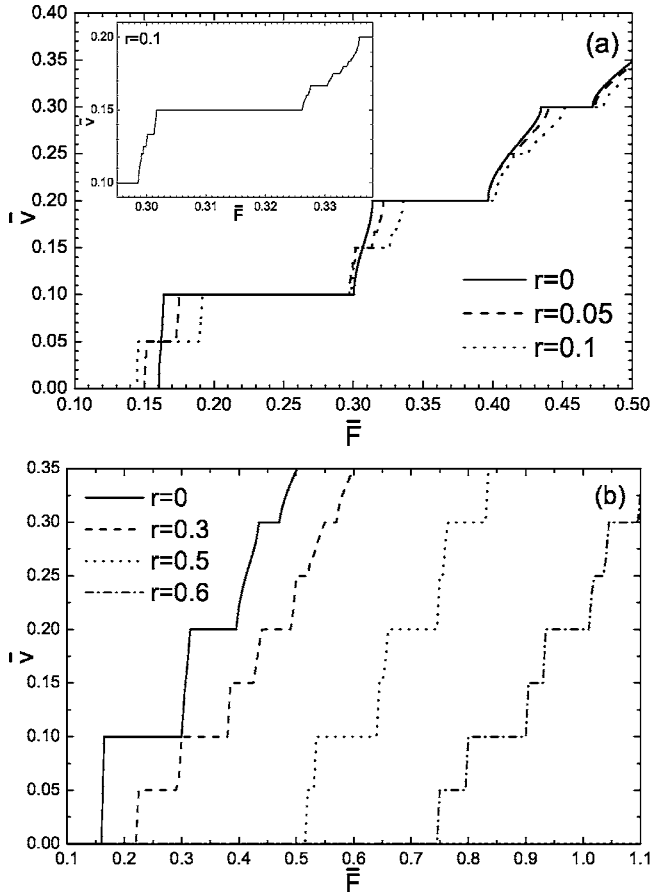

FIG. 2. Average velocity as a function of the average driving force for $\omega=\frac{1}{2}$, $K=4$, $F_{ac}=0.2$, $\nu_0=0.2$, and different values of the shape parameter r: (a) $r\leqslant 0.1$ and (b) $r\geqslant 0.3$. The inset shows the subharmonic steps for $r=0.1$.

The appearance of new steps is a consequence of the deformation of the substrate potential when $r\neq 0$. While for $r=0$, the potential was sinusoidal with a period of 2π and for $\omega=\frac{1}{2}$, inside one period we have two atoms in one potential well; for $r\neq 0$, the period changes to 4π and we have four atoms in two potential wells inside one period. Since the average velocity of the unlocked trajectory is $|\bar{v}-\bar{v}_{step}|=\nu_0/qm$ [q is the number of particles in the unit cell ($\omega=p/q$) and m is an integer], then changing of the number of particles inside the unit cell will change $\bar{v}$ and, consequently, the critical force ($|\bar{v}-\bar{v}_{step}|\approx|F-F_c|^{1/2}$) [2]. In our case, the number of particles in the unit cell changes from two to four, and as we can clearly see from Figs. 2, the new steps appear at the half values of the previous ones. Besides these half integer steps, in the inset of Fig. 2(a), the whole series of subharmonic steps that result from the changing of oscillation frequencies of particles are shown.

The shape of substrate potential has a strong influence on critical depinning force as we can also see in Figs. 2(a) and particularly 2(b). In Fig. 3, the critical force F_c as a function of the shape parameter r for $K=4$, $F_{ac}=0.2$, and $\nu_0=0.2$ is shown.

The observed decrease of F_c in Fig. 2(a) is clearly seen in the inset of Fig. 3. This interesting behavior of F_c is a result

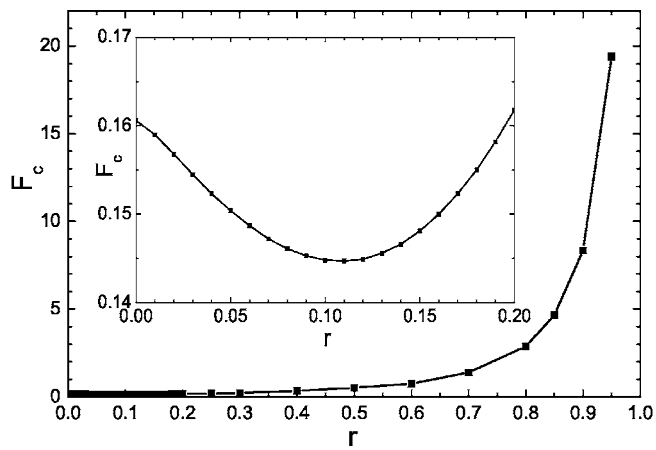

FIG. 3. Critical force as a function of the shape parameter r for $\omega=\frac{1}{2}$, $K=4$, $F_{ac}=0.2$, and $\nu_0=0.2$. The inset shows the same curve in region $r\leqslant 0.2$.

of the competition of two effects. One is the decrease of the critical force due to changing of q and also due to changing of the oscillation frequencies of particles inside the potential minima. Namely, in the sinusoidal potential, the oscillation frequencies of particles in two successive minima are the same. On the contrary, in ASDP, the oscillation frequency of the particles in sharp minima increases while one of the particles in wide minima decreases. As r starts to increase from zero, the particle in minima that is getting sharper is becoming more and more unstable. The system will not move as unique and this could affect the critical force and the steps. Another effect, opposite to the previous one is the increase of F_c due to the increasing discreteness and pinning of the system with increasing of r [8]. If K is not too large, for small values of the shape parameter, in our case for $K=4$, in the region $0<r<0.1$, the critical force can even decrease with increasing r keeping its value lower respect to the value at $r=0$ in the interval $0<r<0.2$. For the larger values of the shape parameter, the second effect dominates and $F_c(r)$ increases with r diverging to the infinity when $r\rightarrow 1$. We made simulations also for the higher value of the pinning parameter $K=6$ and obtained continuously increasing F_c in the whole region of r (only for $r=0.01$ and 0.02, a very small decrease, less than 1% of the critical force was observed).

If ω is an integer, for the standard FK model, the subharmonic steps are absent [6,7,10]. However, when the shape of the substrate potential starts to deviate from the sinusoidal one, even for the integer value of ω, the subharmonic steps appear. In Fig. 4, the response function $\bar{v}(F)$ for the structure with $\omega=1$ and $r=0.5$ is presented.

In the inset of Fig. 4, we can clearly see the whole series of subharmonic steps that appear due to the changing of the number of particles in the unit cell and their oscillation frequencies. This effect represents a novel phenomenon for the commensurate structures with integer values of ω and it cannot be obtained from the standard theories.

The presented results have shown that the shape of the substrate potential plays the crucial role in the dc+ac-driven dynamics of discreet systems. Deformation of the substrate potential from the sinusoidal one changes the number of par-

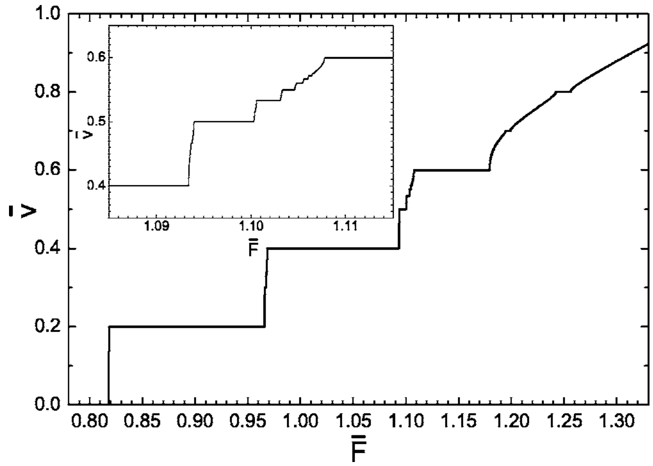

FIG. 4. Average velocity as a function of average driving force for $\omega=1$, $K=4$, $F_{ac}=0.2$, $\nu_0=0.2$, and $r=0.5$. The inset shows the subharmonic steps.

ticles in the unit cell and their oscillation frequencies while, on the other side, the discreteness effects and the pinning increase. Due to the simultaneous influence of these effects, the critical force not only increases with increasing deformation but can even decrease for the small values of shape parameter if the pinning is not too strong. Another very interesting phenomenon is the appearance of subharmonic steps in the commensurate structures with both rational and integer values of the average interparticle distance. On the contrary with the standard FK model that provides subharmonic steps only for the rational values of winding number, in the model with deformable potential, both commensurate systems will exhibit the whole series of subharmonic steps. According to this, we can conclude that the deformation of the substrate potential can generate the subharmonic mode locking in the dynamics of any commensurate structure.

These phenomena could be relevant for the charge-density or spin-density wave systems, vortex lattices, and systems of Josephson-junction arrays. The subharmonic steps have been already experimentally observed [3,5,11] and a matter of many debates. In the charge-density wave systems, they can result from the inertial effects or from the internal degrees of freedom [3,11]. In the Fukayama, Lee, and Rice (FLR) model, they arise from the spatial structure [11]; while in Ref. [7], they are obtained from the few-domain classical model. In the systems with Josephson-junction arrays, they can arise from disorder, inductive effects, or nonsinusoidal current-phase relations [5,12]. In Ref. [13], they are obtained by using free boundary conditions. In the conclusion, the authors even speculate that the periodic lattice might produce subharmonic steps if the potential is not purely sinusoidal. We have considered only one type of deformable potential and examination of other types is necessary, particularly since according to Ref. [14], the half integer and higher order subharmonic steps have different origins, which is what our results also indicate. Besides the above applications, the one very speculative, but also very interesting that could arise from our results is the possibility of an indirect experimental examination of deformations or even intrinsic localized modes [15]. We believe that our results could contribute to the understanding of the origins of subharmonic mode locking in the realistic systems and that they could stimulate new theories and experiments.

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